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EQUIVALENCE OF RENEWAL SEQUENCES AND ISOMORPHISM OF RANDOM WALKS

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ABSTRACT

We show that centred aperiodic random walks on $\mathbb{Z}^d$ whose jump random variables are in $L^2\sqrt{\log^+ L}$ have equivalent renewal sequences. An isomorphism theorem is deduced.

Let X be a $\mathbb{Z}^d$-valued random variable (where $d \in \mathbb{N}$). The **random walk** with **jump random variable** X is a measure preserving transformation T_X of the σ-finite, infinite measure space $(\mathbb{Z}^d)^{\mathbb{Z}} \times \mathbb{Z}^d$ equipped with the σ-algebra generated by cylinder sets, and the measure $f^{\mathbb{Z}}\times$ counting measure (where $f_n =$Prob$(X = n)$). It is defined by

$$T_X\Big((\ldots, x_{-1}, x_0, x_1, \ldots), n\Big) = \Big((\ldots, x_0, x_1, x_2, \ldots), n + x_1\Big).$$

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The random walk is conservative and ergodic in case $d = 1, 2$, and the jump random variable X: has finite second moment ($E(|X|^2) < \infty$), is **centred** ($E(X) = 0$), and is **strictly aperiodic** in the sense that if $\varphi(s) = E(e^{is\cdot X})$ ($s \in \mathbb{R}^d$), then $|\varphi(s)| = 1$ if and only if $s \in 2\pi\mathbb{Z}^d$.

Let $Y^{(d)} = \epsilon \in \{0, \pm1\}^d$ with probability $\frac{1}{3^d}$. It is shown in [1] that if $d = 1, 2$, and X is a $\mathbb{Z}^d$-valued random variable with $E(|X|^7) < \infty$, which is centred, and strictly aperiodic, then T_X and $T_{Y^{(d)}}$ are isomorphic as measure preserving transformations.

We prove here the

MAIN THEOREM: *If $d = 1, 2$, and X is a centred, strictly aperiodic $\mathbb{Z}^d$-valued random variable with $E\left(|X|^2\sqrt{\log^+|X|}\right) < \infty$, then T_X and $T_{Y^{(d)}}$ are isomorphic.*

The method of proof, using results in [1], involves a study of the **equivalence** of renewal sequences of **jump random variables**, that is, renewal sequences of form

$$u(n) = u_n(X) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\varphi(s)^n ds = \text{Prob}\left(\sum_{k=1}^{n} X_k = 0\right)$$

where $X_1, X_2, \ldots$ are i.i.d.r.v.'s, each distributed as X.

Recall from [1] that the renewal sequences u, u' are said to be **equivalent** if there are positively recurrent, aperiodic renewal sequences v, v' such that $u(n)v(n) = u'(n)v'(n)$, and that [1, theorem 3.6] conservative, ergodic random walks whose jump random variables have equivalent renewal sequences are isomorphic.

Let X be a $\mathbb{Z}^d$-valued random variable. A necessary condition for the equivalence of the renewal sequences $u(X)$ and $u(Y^{(d)})$, is that X be centred, strictly aperiodic, and with finite second moment. This is because otherwise, $\liminf_{n\to\infty}\frac{u_n(X)}{u_n(Y^{(d)})} = 0$.

A sufficient condition for equivalence of renewal sequences is given by [1, corollary 4.4], which says that the renewal sequences $u = (u(n))_{n\geq 0}$ and $u(Y^{(d)})$ are equivalent if

$$(*) \qquad \sum_{n\in A} n\left|\log\left(\frac{1}{p_n}\right) - \frac{d}{2n^2}\right| < \infty$$

where

$$p_n = \frac{u(n)^2}{u(n-1)u(n+1)} \quad \text{for } n \in A = \{n \in \mathbb{N}\colon u(k) > 0 \ \forall\, k \geq n-1\}.$$

Thus, given a centred, strictly aperiodic, $\mathbb{Z}^d$-valued jump random variable X, $(d = 1, 2)$, in order to prove that T_X and $T_{Y^{(d)}}$ are isomorphic, it is sufficient to establish that $u(X)$ satisfies $(*)$.

The isomorphism theorem of [1] is proved in this way by means of [1, theorem 5.1] which establishes $(*)$ when $E(|X|^7) < \infty$ and X is centred, strictly aperiodic.

Similarly, we prove our main theorem by establishing the

THEOREM: *Suppose that* $E(|X|^2\sqrt{\log^+|X|}) < \infty$, *and that* X *is centred, and strictly aperiodic, then* $(*)$ *holds.*

Remark: We do not know of any centred, strictly aperiodic random variable X on $\mathbb{Z}^d$ with finite second moment for which $u(X)$ and $u(Y^{(d)})$ are not equivalent.

Proof of the Theorem

It follows from the local limit theorem that

$$\exists \lim_{n\to\infty} n^{d/2}u(n) \in \mathbb{R}_+,$$

and hence that $(*)$ holds if and only if

$$\sum_{n=1}^{\infty} n^{d+1}\left|u(n-1)u(n+1) - u(n)^2 - \frac{du(n)^2}{2n^2}\right| < \infty.$$

Set

$$D_n := n^{d+1}\left(u(n-1)u(n+1) - u(n)^2 - \frac{du(n)^2}{2n^2}\right),$$

and let $a_n \sim a_n'$ mean that $\sum_n |a_n - ca_n'| < \infty$ for some $0 < c < \infty$. In particular, if $a_n \sim a_n'$, then $\sum_n |a_n| < \infty$ iff $\sum_n |a_n'| < \infty$.

We'll show, under the assumption $E(|X|^2\sqrt{\log^+|X|}) < \infty$, that $D_n \sim 0$.

Using Taylor's theorem for φ around 0, we find that,

$$(\ddagger) \qquad \exists\, \delta > 0,\ \epsilon \in (0, \tfrac{1}{2}] \text{ such that } 0 < |\varphi(x)| \leq e^{-\epsilon\Gamma(x)}\ \forall\, |x| \leq \delta$$
$$\text{where } \Gamma(x) := E((X\cdot x)^2) \quad (x \in \mathbb{R}^d).$$

By aperiodicity of X, $\sup_{|t|\geq\delta}|\varphi(t)|<1$, and we have that

$$\begin{aligned}
D_n &:= n^{d+1}\Big(u(n-1)u(n+1)-u(n)^2-\frac{du(n)^2}{2n^2}\Big)\\
&=\frac{n^{d+1}}{(2\pi)^{2d}}\iint_{\mathbb{T}^d\times\mathbb{T}^d}\Big(\varphi(x)^{n-1}\varphi(y)^{n+1}-\varphi(x)^n\varphi(y)^n\big(1+\frac{d}{2n^2}\big)\Big)dxdy\\
&\sim\frac{n^{d+1}}{(2\pi)^{2d}}\iint_{N(0,\delta)^2}\Big(\varphi(x)^{n-1}\varphi(y)^{n+1}-\varphi(x)^n\varphi(y)^n\big(1+\frac{d}{2n^2}\big)\Big)dxdy\\
&=\frac{n^{d+1}}{(2\pi)^{2d}}\iint_{N(0,\delta)^2}\varphi(x)^n\varphi(y)^n\Big(\frac{\varphi(x)-\varphi(y)}{\varphi(y)}-\frac{d}{2n^2}\Big)dxdy\\
&=\frac{n^{d+1}}{(2\pi)^{2d}}\iint_{N(0,\delta)^2}\varphi(x)^n\varphi(y)^n\Big(\frac{\varphi(x)-\varphi(y)}{\varphi(y)}-(\varphi(x)-\varphi(y))-\frac{d}{2n^2}\Big)dxdy
\end{aligned}$$

since

$$\iint_{N(0,\delta)^2}\varphi(x)^n\varphi(y)^n(\varphi(x)-\varphi(y))dxdy=0.$$

Here, $N(0,\delta):=\{x\in\mathbb{T}^d=[-\pi,\pi]^d\colon |x|<\delta\}$.

Changing variables, we have

$$\begin{aligned}
D_n &\sim\frac{n}{(2\pi)^{2d}}\iint_{N(0,\delta\sqrt{n})^2}\varphi\big(\frac{s}{\sqrt{n}}\big)^n\varphi\big(\frac{t}{\sqrt{n}}\big)^n\\
&\qquad\cdot\Bigg(\frac{[\varphi(\frac{s}{\sqrt{n}})-\varphi(\frac{t}{\sqrt{n}})](1-\varphi(\frac{t}{\sqrt{n}}))}{\varphi(\frac{t}{\sqrt{n}})}-\frac{d}{2n^2}\Bigg)dsdt\\
&=\frac{1}{n(2\pi)^{2d}}\iint_{N(0,\delta\sqrt{n})^2}\varphi\big(\frac{s}{\sqrt{n}}\big)^n\varphi\big(\frac{t}{\sqrt{n}}\big)^n\\
&\qquad\cdot\Bigg(\frac{(\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}}))\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})}{\varphi(\frac{t}{\sqrt{n}})}-\frac{d}{2}\Bigg)dsdt,
\end{aligned}$$

where $\Gamma(x):=E((X\cdot x)^2)=x^*\Gamma x\ \ (x\in\mathbb{R}^d)$, and $1-\varphi(s)=\frac{\Gamma(s)}{2}\phi(s)$.

Using (‡), and $1-\varphi(t/\sqrt{n})=O(|t|^2/n)$, we have, by Lebesgue's dominated convergence theorem, that

$$\begin{aligned}
D_n &\sim\frac{1}{n}\iint_{N(0,\delta\sqrt{n})^2}\varphi\big(\frac{s}{\sqrt{n}}\big)^n\varphi\big(\frac{t}{\sqrt{n}}\big)^n\\
&\qquad\cdot\Big(\big[\frac{\Gamma(t)}{2}\phi\big(\frac{t}{\sqrt{n}}\big)-\frac{\Gamma(s)}{2}\phi\big(\frac{s}{\sqrt{n}}\big)\big]\frac{\Gamma(t)}{2}\phi\big(\frac{t}{\sqrt{n}}\big)-\frac{d}{2}\Big)dsdt.
\end{aligned}$$

Write

$$a_n(s)=\frac{\Gamma(s)}{2}\Big(1-\phi\big(\frac{s}{\sqrt{n}}\big)\Big)=n\Big(\varphi\big(\frac{s}{\sqrt{n}}\big)-1+\frac{\Gamma(s)}{2n}\Big).$$

Using again the Taylor expansion of φ around 0, we obtain $K \in \mathbb{R}_+$ such that

$$|a_n(s)| \le KE\Big((s\cdot X)^2\Big(\frac{|X||s|}{\sqrt{n}} \wedge 1\Big)\Big) \le K\Gamma(s) \quad \forall\, s \in \mathbb{R}^d, \quad n \in \mathbb{N}.$$

To continue, we need to use the moment assumption on X.

LEMMA 1: *If* $E\Big(|X|^2(\log^+|X|)^{\frac{1}{k}}\Big) < \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{|a_n(s)|^k}{n} \le M_k|s|^{2k}(1+|s|)^k,$$

where $M_k \in \mathbb{R}_+$, *and hence*

$$\int_{\mathbb{R}^d} |s|^\ell \Big(\sum_{n=1}^{\infty} \frac{|a_n(s)|^k}{n}\Big) e^{-\epsilon\Gamma(s)} ds < \infty \quad \forall\, \ell \ge 0.$$

Proof:

$$\begin{aligned}|a_n(s)| &\le KE\Big((s\cdot X)^2\big(\frac{|X||s|}{\sqrt{n}} \wedge 1\big)\Big)\\ &\le \frac{K|s|^2(1+|s|)}{\sqrt{n}} E(|X|^2(|X| \wedge \sqrt{n})),\end{aligned}$$

$$\therefore \quad \frac{|a_n(s)|^k}{n} \le \frac{K^k|s|^{2k}(1+|s|)^k}{n^{1+\frac{k}{2}}}\Big(E(|X|^2(|X| \wedge \sqrt{n}))\Big)^k$$

and it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{k}{2}}}(E(|X|^2(|X| \wedge \sqrt{n})))^k < \infty.$$

Now,

$$E(|X|^2(|X| \wedge \sqrt{n})) = E(|X|^3 1_{[|X| \le \sqrt{n}]}) + \sqrt{n}E(|X|^2 1_{[|X| \ge \sqrt{n}]}).$$

Therefore, by Jensen's inequality,

$$\begin{aligned}\Big(E(|X|^2(|X| \wedge \sqrt{n}))\Big)^k \le & 2^{k-1}\Big(E(|X|^3 1_{[|X| \le \sqrt{n}]})\Big)^k\\ & + 2^{k-1}n^{\frac{k}{2}}\Big(E(|X|^2 1_{[|X| \ge \sqrt{n}]})\Big)^k,\end{aligned}$$

and it is now sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{k}{2}}}(E(|X|^3 1_{[|X|\le\sqrt{n}]})\Big)^k<\infty \quad \& \quad \sum_{n=1}^{\infty}\frac{1}{n}\Big(E(|X|^2 1_{[|X|\ge\sqrt{n}]})\Big)^k<\infty.$$

Letting $X_1,\dots,X_k$ be independent copies of X, we have

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{1}{n^{1+\frac{k}{2}}}\Big(E(|X|^3 1_{[|X|\le\sqrt{n}]})\Big)^k\\ &=E\Big(|X_1|^3\dots|X_k|^3\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{k}{2}}}1_{[|X_1|\le\sqrt{n}]}\cdots 1_{[|X_k|\le\sqrt{n}]}\Big)\\ &=E\Big(|X_1|^3\dots|X_k|^3\sum_{n\ge|X_1|^2\vee\cdots\vee|X_k|^2}\frac{1}{n^{1+\frac{k}{2}}}\Big)\\ &\le 2E\Big(\frac{|X_1|^3\dots|X_k|^3}{(|X_1|\vee\cdots\vee|X_k|)^k}\Big)\\ &\le 2\Big(E(|X|^2)\Big)^k<\infty,\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\Big(E(|X|^2 1_{[|X|\ge\sqrt{n}]})\Big)^k&=E\Big(|X_1|^2\dots|X_k|^2\sum_{n=1}^{\infty}\frac{1}{n}1_{[|X_1|\ge\sqrt{n}]}\cdots 1_{[|X_k|\ge\sqrt{n}]}\Big)\\ &=E\Big(|X_1|^2\dots|X_k|^2\sum_{n=1}^{|X_1|^2\wedge\cdots\wedge|X_k|^2}\frac{1}{n}\Big)\\ &\le E\Big(|X_1|^2\dots|X_k|^2\log^+(|X_1|^2\wedge\cdots\wedge|X_k|^2)\Big)\\ &\le\Big(E(|X|^2(\log^+|X|^2)^{\frac{1}{k}})\Big)^k<\infty.\end{aligned}$$

$$\therefore\quad \sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{k}{2}}}(E(|X|^2(|X|\wedge\sqrt{n})))^k<\infty. \qquad \blacksquare$$

LEMMA 2: *If* $E(|X|^2\sqrt{\log^+|X|})<\infty$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{N(0,\delta\sqrt{n})}|s|^{\ell}|\varphi(\frac{s}{\sqrt{n}})^n-e^{-\Gamma(s)/2}(1+a_n(s))|ds<\infty\quad\forall\,\ell\ge 0.$$

Proof: Using Taylor's theorem yet again, we have that $\exists\, M \in \mathbb{R}_+$ such that

$$\varphi\Big(\frac{s}{\sqrt{n}}\Big) - e^{-\frac{\Gamma(s)}{2n}} = \frac{a_n(s)}{n} + \Theta_n(s) \quad \forall\, s \in \mathbb{R}$$

where $|\Theta_n(s)| \le M|s|^4/n^2$.

Therefore,

$$\begin{aligned}\varphi\Big(\frac{s}{\sqrt{n}}\Big)^n - e^{-\frac{\Gamma(s)}{2}} &= \sum_{k=0}^{n-1} \varphi\Big(\frac{s}{\sqrt{n}}\Big)^{n-1-k} e^{-\frac{k\Gamma(s)}{2n}} \Big(\frac{a_n(s)}{n} + \Theta_n(s)\Big) \\ &= \frac{a_n(s)}{n} \sum_{k=0}^{n-1} \varphi\Big(\frac{s}{\sqrt{n}}\Big)^{n-1-k} e^{-\frac{k\Gamma(s)}{2n}} + \widetilde{\Theta}_n(s),\end{aligned}$$

where

$$|\widetilde{\Theta}_n(s)| \le n|\Theta_n(s)| e^{-\epsilon(1-\frac{1}{n})\Gamma(s)} \le \frac{M|s|^4}{n} e^{-\frac{\epsilon}{2}\Gamma(s)} \quad \forall\, |s| \le \delta\sqrt{n}.$$

Proceeding further,

$$\begin{aligned}&\Big(\varphi\Big(\frac{s}{\sqrt{n}}\Big)^n - e^{-\frac{\Gamma(s)}{2}}\big(1 + e^{\frac{\Gamma(s)}{2n}} a_n(s)\big)\Big) \\ &\qquad = \frac{a_n(s)}{n} \sum_{k=0}^{n-1} e^{-\frac{(n-1-k)\Gamma(s)}{2n}} \Big(\varphi\Big(\frac{s}{\sqrt{n}}\Big)^k - e^{-\frac{k\Gamma(s)}{2n}}\Big) + \widetilde{\Theta}_n(s).\end{aligned}$$

Now

$$\Big(\varphi\Big(\frac{s}{\sqrt{n}}\Big)^k - e^{-\frac{k\Gamma(s)}{2n}}\Big) = \sum_{\nu=0}^{k-1} \varphi\Big(\frac{s}{\sqrt{n}}\Big)^{\nu} e^{-\frac{(k-\nu-1)\Gamma(s)}{2n}} \Big(\frac{a_n(s)}{n} + \Theta_n(s)\Big),$$

and hence

$$\Big|\varphi\Big(\frac{s}{\sqrt{n}}\Big)^k - e^{-\frac{k\Gamma(s)}{2n}}\Big| \le k e^{-\frac{\epsilon(k-1)\Gamma(s)}{n}} \Big(\frac{|a_n(s)|}{n} + \Theta_n(s)\Big).$$

Substituting back in the above, we obtain for $|s| \le \delta\sqrt{n}$, $n \ge 4$,

$$\begin{aligned}&\Big|\varphi\Big(\frac{s}{\sqrt{n}}\Big)^n - e^{-\frac{\Gamma(s)}{2}}\big(1 + e^{\frac{\Gamma(s)}{2n}} a_n(s)\big)\Big| \\ &\le \frac{|a_n(s)|}{n} \sum_{k=0}^{n-1} e^{-\frac{(n-1-k)\Gamma(s)}{2n}} \Big(k e^{-\frac{(k-1)\epsilon\Gamma(s)}{n}} \Big(\frac{|a_n(s)|}{n} + |\Theta_n(s)|\Big)\Big) + |\widetilde{\Theta}_n(s)| \\ &\le e^{-\frac{\epsilon\Gamma(s)}{2}} \Big(|a_n(s)|^2 + n|a_n(s)||\Theta_n(s)|\Big) + |\widetilde{\Theta}_n(s)| \\ &\le e^{-\frac{\epsilon\Gamma(s)}{2}} \Big(|a_n(s)|^2 + KME(|X|^2)|\frac{|s|^6}{n} + \frac{M|s|^4}{n}\Big).\end{aligned}$$

Lastly, by Taylor's theorem, $\exists\ \overline{M} \in \mathbb{R}_+$ such that

$$0 < e^{\frac{\Gamma(s)}{2n}} - 1 \le \frac{\overline{M}}{n} \quad \forall\, n \ge 1, \quad |s| < \delta\sqrt{n},$$

whence for $|s| < \delta\sqrt{n}$,

$$\begin{aligned}&\left|\varphi\Big(\frac{s}{\sqrt{n}}\Big)^n - e^{-\frac{\Gamma(s)}{2}}(1 + a_n(s))\right| \\ &\le e^{-\frac{\epsilon\Gamma(s)}{2}}\left(|a_n(s)|^2 + KME(|X|^2)|\frac{|s|^6}{n} + \frac{M|s|^4}{n} + \overline{M}K\frac{\Gamma(s)}{n}\right),\end{aligned}$$

and Lemma 2 is established by Lemma 1. ∎

As a corollary of Lemma 2, we get that

$$\begin{aligned}D_n \sim n^{-1}\iint_{N^2} & e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}(1 + a_n(s))(1 + a_n(t)) \\ &\cdot\left(\Big(\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{\Gamma(s)}{2}\phi\Big(\frac{s}{\sqrt{n}}\Big)\Big)\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{d}{2}\right)dsdt\end{aligned}$$

where $N = N(0, \delta\sqrt{n})$. This is because

$$\left|\Big(\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{\Gamma(s)}{2}\phi\Big(\frac{s}{\sqrt{n}}\Big)\Big)\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{d}{2}\right| \le f(s,t)$$
$$\forall\, n \ge 1, \quad |s|, |t| < \delta\sqrt{n},$$

where f is a polynomial in $|s|$ and $|t|$.

Using the existence of $0 < r_k < 1$ ($k \ge 1$ fixed), such that

$$\int_{\mathbb{R}^d \setminus N(0,\delta\sqrt{n})} |s|^k e^{-\frac{\Gamma(s)}{2}} = O(r_k^n) \quad \text{as } n \to \infty,$$

we obtain that

$$\begin{aligned}D_n \sim n^{-1}\iint & e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}(1 + a_n(s))(1 + a_n(t)) \\ &\cdot\left(\Big(\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{\Gamma(s)}{2}\phi\Big(\frac{s}{\sqrt{n}}\Big)\Big)\frac{\Gamma(t)}{2}\phi\Big(\frac{t}{\sqrt{n}}\Big) - \frac{d}{2}\right)dsdt\end{aligned}$$

where here, and henceforth, we supress the domain of integration where this is maximal.

Our next task is to simplify our expression $D_n' \sim D_n$ on the basis of the identity

$$(\dagger) \qquad \iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}}\left(\Big(\frac{\Gamma(t)}{2} - \frac{\Gamma(s)}{2}\Big)\frac{\Gamma(t)}{2} - \frac{d}{2}\right)dsdt = 0.$$

LEMMA 3: *If* $E(|X|^2\sqrt{\log^+|X|}) < \infty$, *then*

$$(2) \qquad D_n \sim \frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}} a_n(s)\Big(\Gamma(s)^2 - 2(d+2)\Gamma(s) + d(d+2)\Big)ds.$$

Proof: It will be convenient to use the following notation:

$$f_n(s,t) \approx g_n(s,t) \text{ if } \frac{1}{n}\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}} f_n(s,t)dsdt \sim \frac{1}{n}\iint e^{-\frac{\Gamma(s)+\Gamma(t)}{2}} g_n(s,t)dsdt.$$

We'll use the following consequence of Lemma 1,

$$\text{(i)} \qquad a_n(s)^2 \approx a_n(s)a_n(t) \approx 0.$$

Also, by symmetry,

$$\text{(ii)} \qquad f_n(s,t) \approx g_n(s,t) := f_n(t,s).$$

We'll also use the following formulae:

$$\text{(iii)} \qquad \int_{\mathbb{R}^d} \Phi(\Gamma(s))e^{-\frac{\Gamma(s)}{2}}ds = \frac{1}{\sqrt{\det\Gamma}}\int_{\mathbb{R}^d}\Phi(|s|^2)e^{-\frac{|s|^2}{2}}ds,$$

$$\text{(iv)} \qquad \int_{\mathbb{R}^d} e^{-\frac{|s|^2}{2}}|s|^{2k}ds = (2\pi)^{\frac{d}{2}}r_{2k}$$

where $r_0 = 1$, $r_2 = d$, $r_4 = d(d+2)$, and $r_6 = d(d+2)(d+4)$,

$$
\begin{aligned}
&(1+a_n(s))(1+a_n(t))\Bigg(\Big(\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{\Gamma(s)}{2}\phi(\frac{s}{\sqrt{n}})\Big)\frac{\Gamma(t)}{2}\phi(\frac{t}{\sqrt{n}})-\frac{d}{2}\Bigg)\\
&=(1+a_n(s))(1+a_n(t))\Bigg(\big[\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2}-(a_n(t)-a_n(s))\big]\big[\frac{\Gamma(t)}{2}-a_n(t)\big]-\frac{d}{2}\Bigg)\\
&\overset{t}{\approx}\Bigg(\{\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2}-[a_n(t)-a_n(s)]\}\{\frac{\Gamma(t)}{2}-a_n(t)\}-\{\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2}\}\frac{\Gamma(t)}{2}\Bigg)\\
&\qquad+[a_n(s)+a_n(t)]\Bigg(\Big(\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2}\Big)\frac{\Gamma(t)}{2}-\frac{d}{2}\Bigg)\overset{(i)}{\approx}\\
&\qquad-a_n(t)(\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2})-[a_n(t)-a_n(s)]\frac{\Gamma(t)}{2}\\
&\qquad+[a_n(s)+a_n(t)]\Bigg((\frac{\Gamma(t)}{2}-\frac{\Gamma(s)}{2})\frac{\Gamma(t)}{2}-\frac{d}{2}\Bigg)\\
&=a_n(s)\Bigg(\frac{\Gamma(t)^2}{4}-\frac{\Gamma(s)\Gamma(t)}{4}+\frac{\Gamma(t)}{2}-\frac{d}{2}\Bigg)\\
&\qquad+a_n(t)\Bigg(\frac{\Gamma(t)^2}{4}-\frac{\Gamma(s)\Gamma(t)}{4}+\frac{\Gamma(s)}{2}-\Gamma(t)-\frac{d}{2}\Bigg)\\
&\overset{(ii)}{\approx}a_n(s)\Bigg(\frac{\Gamma(s)^2}{4}+\frac{\Gamma(t)^2}{4}-\frac{\Gamma(s)\Gamma(t)}{2}-\Gamma(s)+\Gamma(t)-d\Bigg)\\
&\overset{(iii)}{\approx}a_n(s)\Bigg(\frac{\Gamma(s)^2}{4}r_0+\frac{r_4}{4}-\frac{r_2\Gamma(s)}{2}-\Gamma(s)r_0+r_2-dr_0\Bigg)\\
&\overset{(iv)}{=}a_n(s)\Bigg(\frac{\Gamma(s)^2}{4}+\frac{d(d+2)}{4}-\frac{d\Gamma(s)}{2}-\Gamma(s)\Bigg)\\
&=\frac{1}{4}a_n(s)\Bigg(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Bigg). \qquad \blacksquare
\end{aligned}
$$

Remark: It now follows from Lemmas 1 and 3 that (∗) holds if, in addition, $E(|X|^2\log^+|X|)<\infty$.

To continue, we calculate

$$\frac{1}{n}\int e^{-\Gamma(s)/2}a_n(s)\Bigg(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Bigg)ds$$

more precisely using the formulae

(v) $$\int_{\mathbb{R}^d}e^{-|s|^2/2}|s|^{2k}e^{i\gamma\cdot s}ds=(2\pi)^{d/2}q_{2k}(\gamma)e^{-|\gamma|^2/2}$$

(vi) where $q_0=1,\ q_2(\gamma)=d-|\gamma|^2,\ q_4(\gamma)=|\gamma|^4-2(d+2)|\gamma|^2+d(d+2)$.

LEMMA 4:

$$\frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)ds = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{|\det\Gamma|}}E\Big(e^{-\frac{|A^{-1*}X|^2}{2n}}\frac{|A^{-1*}X|^4}{n^2}\Big). \tag{3}$$

Proof:

$$\frac{1}{n}\int e^{-\frac{\Gamma(s)}{2}}a_n(s)\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)ds$$
$$=\int e^{-\frac{\Gamma(s)}{2}}\Big(\Gamma(s)^2-2(d+2)\Gamma(s)+d(d+2)\Big)\Big(\varphi(\frac{s}{\sqrt{n}})-1+\frac{\Gamma(s)}{2n}\Big)ds$$
$$\overset{\text{(iii)}}{=}\frac{1}{\sqrt{|\det\Gamma|}}\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))(\varphi(\frac{A^{-1}x}{\sqrt{n}})-1+\frac{|x|^2}{2n})dx$$

where $\Gamma=A^*A$. Next,

$$\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))\varphi\Big(\frac{A^{-1}x}{\sqrt{n}}\Big)dx$$
$$=E\Big(\int e^{-\frac{|x|^2}{2}}e^{i\frac{A^{-1}x\cdot X}{\sqrt{n}}}(|x|^4-2(d+2)|x|^2+d(d+2))dx\Big)$$
$$\overset{\text{(v)}}{=}(2\pi)^{\frac{d}{2}}E\Big(e^{-\frac{|A^{-1*}X|^2}{2n}}\Big(q_4\Big(\frac{A^{-1*}X}{\sqrt{n}}\Big)$$
$$-2(d+2)q_2\Big(\frac{A^{-1*}X}{\sqrt{n}}\Big)+d(d+2)q_0\Big(\frac{A^{-1*}X}{\sqrt{n}}\Big)\Big)\Big)$$
$$\overset{\text{(vi)}}{=}(2\pi)^{\frac{d}{2}}E\Big(e^{-|A^{-1*}X|^2/2n}\frac{|A^{-1*}X|^4}{n^2}\Big).$$

$$\int e^{-\frac{|x|^2}{2}}(|x|^4-2(d+2)|x|^2+d(d+2))\Big(-1+\frac{|x|^2}{2n}\Big)dx$$
$$=\int e^{-\frac{|x|^2}{2}}\Big[\frac{|x|^6}{2n}-\Big(1+\frac{d+2}{n}\Big)|x|^4+\Big(2(d+2)+\frac{d(d+2)}{2n}\Big)|x|^2-d(d+2)\Big]dx$$
$$\overset{\text{(iv)}}{=}0. \qquad ∎$$

We have, by (2) and (3), that

$$D_n\sim E\Big(e^{-|A^{-1*}X|^2/2n}\frac{|A^{-1*}X|^4}{n^2}\Big).$$

To conclude the proof of the theorem,

$$\begin{aligned}
\sum_{n=1}^{\infty} E\left(\frac{|A^{-1*}X|^4}{n^2} e^{-\frac{|A^{-1*}X|^2}{2n}}\right) &= \sum_{n=1}^{\infty} E\left(\frac{|A^{-1*}X|^4}{n^2} \int_{\frac{|A^{-1*}X|}{\sqrt{n}}}^{\infty} se^{-\frac{s^2}{2}} ds\right) \\
&= \int_0^{\infty} se^{-\frac{s^2}{2}} E\left(\sum_{n=1}^{\infty} \frac{|A^{-1*}X|^4}{n^2} 1_{[|A^{-1*}X| \le s\sqrt{n}]}\right) ds \\
&\le 2\int_0^{\infty} se^{-\frac{s^2}{2}} E\left(|A^{-1*}X|^2 (s^2 \wedge |A^{-1*}X|^2)\right) ds \\
&\le 2E(|A^{-1*}X|^2) \int_0^{\infty} s^3 e^{-\frac{s^2}{2}} ds \\
&< \infty. \quad \blacksquare
\end{aligned}$$

References

[1] J. Aaronson and M. Keane, *Isomorphism of random walks*, Israel J. Math., this issue.